Imp. (M.T.U. 2014)

Question: Find the moment generating function of the exponential distribution $f(x) = \frac{1}{c} e^{-x/c}$, $0 \le x \le \infty$, $c > 0$

Hence find its mean and standard deviation.

Solution: Moment generating function about the origin is given by

$$M_x(t) = \int_0^\infty e^{tx} \cdot \frac{1}{c} e^{-x/c} dx$$

$$= \frac{1}{c}\int_0^\infty e^{\left(t-\frac{1}{c}\right)x} dx = \frac{1}{c}\left[\frac{e^{\left(t-\frac{1}{c}\right)x}}{\left(t-\frac{1}{c}\right)}\right]_0^\infty$$

$$= (1-ct)^{-1} = 1 + ct + c^2t^2 + c^3t^3 + \ldots\ldots$$

$$\therefore v_1 = \left[\frac{d}{dt} M_x(t)\right]_{t=0} = \left[c + 2c^2t + 3c^3t^2 + \cdots\right]_{t=0} = c$$

and $v_2 = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = 2c^2$

Now, mean $\bar{x} = v_1 = c$

Variance $= (\text{Standard deviation})^2 = \mu_2 = v_2 - \bar{x}^2$

$$\mu_2 = v_2 - v_1^2 = 2c^2 - c^2 = c^2$$

$$\boxed{\therefore \text{S.D.} = \sqrt{\mu_2} = c.}$$

Question: Obtain the moment generating function of the random variable x having probability distribution

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 \le x < 2 \\ 0 & \text{for elsewhere.} \end{cases}$$

Also determine mean v_1, v_2 and variance μ_2.

Solution: $M_x(t) = E(e^{tx})$

$$= \int_0^1 x e^{tx} dx + \int_1^2 (2-x) e^{tx} dx + \int_2^\infty 0 \cdot e^{tx} dx$$

$$= \left(\frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2}\right)_0^1 + \left(2\frac{e^{tx}}{t} - \frac{xe^{tx}}{t} + \frac{e^{tx}}{t^2}\right)_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \left[\left(2\frac{e^{2t}}{t} - 2\frac{e^{2t}}{t} + \frac{e^{2t}}{t^2}\right]\right.$$

$$\left. - \left(\frac{2e^t}{t} - \frac{e^t}{t} + \frac{e^t}{t^2}\right)\right]$$

$$= \frac{e^{2t} - 2e^t + 1}{t^2} = \left(\frac{e^t - 1}{t}\right)^2$$

Now

$$\left(\frac{e^t - 1}{t}\right)^2 = \frac{\left[1 + t + \frac{t^2}{\lfloor 2} + \frac{t^3}{\lfloor 3} + \text{----} - 1\right]^2}{t^2}$$

$$M_x(t) = 1 + t + t^2 + \text{------}$$

$$\text{Mean} = v_1 = \left[\frac{d}{dt} M_x(t)\right]_{t=0} = 1$$

$$v_2 = \left[\frac{d^2}{dt^2} M_x(t)\right] = 2$$

$$v_3 = \left[\frac{d^3}{dt^3} M_x(t)\right] = \text{------}$$

Similarly, $v_2 = 2$, $\mu_2 = v_2 - \bar{x}^2$

$$= v_2 - v_1^2 = 2 - (1)^2 = 1$$

$$\boxed{\mu_2 = \text{Variance} = 1 \quad \text{Ans.}}$$